Tutorial 3: The Discrete Fourier Transform (DFT) and the FFT

1. DTFT and the DFT

Since most of the signals of practical interest are either finite-length or only a finite set of samples of the signal is available, we will only consider the computation of the spectra of finite-length signals. The discrete-time Fourier transform (DTFT) of a finite-length sequence \( \{x(n), 0 \leq n \leq N - 1\} \) is given by

\[
X(e^{jw}) = \sum_{n=0}^{N-1} x(n)e^{-jwn} \tag{1}
\]

The problem with the calculation of the DTFT is that the frequency \( w \) is continuous, and as such it is not computationally possible. Thus, it is desirable to discretize the frequency (i.e., sample \( X(e^{jw}) \)). The duality of the Fourier representation makes it possible to apply the sampling theorem to continuous spectra in frequency. The limiting conditions are then imposed on the signal in the time domain, i.e., the signal must be time-limited or finite-length. So we can sample (1) to get the discrete Fourier transform (DFT):

\[
X(k) = X(e^{jw}) \bigg|_{w=2\pi k/L} = \sum_{n=0}^{L-1} x(n)e^{-j2\pi nk/L} \tag{2}
\]

where \( x(n) = 0, N \leq n \leq L - 1 \). The effect of sampling in the frequency domain is that the finite-length signal is repeated periodically. According to the sampling theorem, then if we sample the spectrum every \( 2\pi/L \) radians, then \( x(n) \), including the padding zeros, must be repeated periodically every \( L \) sample. To insure there are no overlaps we need to impose the condition that \( L \geq N \). Once the spectrum is sampled correctly then we have that both the time and the frequency representations are periodic. This is the reason why given a finite-length sequence to calculate its DFT we generated a periodic signal of period \( L \geq N \).

Equation (2) is the DFT of \( x(n) \), the inverse discrete Fourier transform (IDFT) is given by

\[
x(n) = \sum_{k=0}^{L-1} X(k)e^{j2\pi nk/L} \tag{3}
\]

Which can be easily obtained from (2) using the orthogonality of the Fourier basis \( \{e^{j2\pi nk/L}\} \).

1.1. Implementation of DFT with Fast Fourier Transform (FFT)

An efficient way to calculate the DFT is the FFT algorithm. The FFT should not be taken as a new transformation, but as a fast way to calculate the DFT. The mechanics of using the FFT to calculate the DFT of a signal \( x(n) \) is the following:

1. If \( x(n) \) is periodic of period \( N \) continue, otherwise go to step 2
• Set the length of the FFT to \( L = N \) (or a multiple of \( N \) for a better frequency resolution)

• Calculate the FFT of a period (or of several periods). The result is

\[ X(k), k = 0, \ldots, L - 1 \]

which is in general complex.

• The magnitude and phase spectra are given by \(|X(k)|\) and \(\arg[X(k)]\) for \( k = 0, \ldots, L - 1 \), respectively. If using more than one period, say \( M \), then divide the magnitude values by \( M \) (see Remarks below for the reason).

2. If the signal is aperiodic of length \( N \), then

• Generate a periodic signal \( \tilde{x}(n) \), of period \( L \geq N \) (padding with zeros from \( N \) to \( L - 1 \) is needed)

• Calculate the FFT of a period, i.e., \( \tilde{x}(n), 0 \leq n \leq L - 1 \)

• Calculate the magnitude and phase of the \( X(k) \) and plot them versus \( w \) (or \( k = 0, \ldots, L - 1 \) to obtain the magnitude and phase spectra.

**Examples**

1. Suppose you want to calculate the DFT of an aperiodic signal of length \( N = 2 \),

\[
x(n) = \begin{cases} 1 & 0 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

Let us generate a periodic signal \( \tilde{x}(n) \) of period \( L = 3, L > N \), then we have that

\[
X_1(k) = \sum_{n=0}^{2} \tilde{x}(n)e^{-j\frac{2\pi}{3}nk} \\
= 1 + e^{-j\frac{2\pi}{3}k} = 2\cos(\pi k/3)e^{-j\pi k/3} \quad 0 \leq k \leq 2
\]

which is the result obtained by calculating the DTFT of \( x(n) \) and then sampling it, i.e.,

\[
X_1(k) = X(e^{jw}) \bigg|_{w=\frac{2\pi k}{3}} \quad 0 \leq k \leq 2
\]

2. Suppose then that we choose \( L = 1, L < N \), then \( X_2(k) = 1 \), which certainly does not equal the DTFT sampled.

3. Now take \( L = 6 \). To generate the periodic signal \( \tilde{x}(n) \) from \( x(n) \) we need to add two zeros, so that

\[
X_3(k) = \sum_{n=0}^{5} \tilde{x}(n)e^{-j\frac{2\pi nk}{6}} \\
= 1 + e^{-j\frac{2\pi k}{6}} \quad 0 \leq k \leq 5
\]
We then have that $X_3(0) = X_1(0) = 2$, $X_3(2) = X_1(1)$ and $X_3(4) = X_1(2)$ and $X_3(1), X_3(3)$, and $X_3(5)$ are new values not available from before. We have increased the number of samples and we have that some coincide with the ones calculated in item 1 above. Adding extra zeros increases the density of the spectral plots, i.e., improves the frequency resolution.

4. Consider $x(n)$ given by

$$x(n) = \begin{cases} 1 & 0 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Let $L = 64$, doing the calculations by computer we obtain the results shown in Figs. 1 and 2. Notice that the phase is not continuous due to the existence of zeros on the unit circle (at frequencies around 1, 2.1 and $\pi$ radians as shown in the magnitude spectrum).

Remarks

- The frequency resolution of the DFT of a periodic signal is improved by considering several periods. Suppose we calculate the DFT of 2 periods of $x(n)$, periodic of period $N$:

$$X_1(k) = \sum_{n=0}^{2N-1} x(n)e^{-j\frac{2\pi}{2N}nk}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}n\frac{k}{2}} + \sum_{n=N}^{2N-1} x(n)e^{-j\frac{2\pi}{N}n\frac{k}{2}}$$

$$= X(k/2) + \sum_{\mu=0}^{N-1} x(\mu + N)e^{-j\frac{2\pi}{N}(\mu+N)\frac{k}{2}}$$

$$= X(k/2) + X(k/2)e^{-j\pi k}$$

$$= (1 + (-1)^k)X(k/2)$$

which gives $X_1(k) = 2X(k/2)$ when $k = 0, 2, \cdots, 2(N - 1)$ (i.e., even), and $X_1(k) = 0$ when $k = 1, 3, \cdots, 2N - 1$. We thus get a scaled version of the spectra calculated when $L = N$. More important, however, is the fact that the frequency resolution is improved and in this case it accentuates the harmonic components.

Example

Consider the FFT of a periodic signal $x(n)$, of period 8, shown in Fig. 3, and for which we calculate its FFT using lengths 8 and 16. The results are shown in Figs. 4-7. Notice that when we increase the number of periods, we get the same number of harmonic components as before, but the number of zeros in between is increased. Also notice that when we use two periods in the calculation of the FFT for the periodic signal in Fig. 3, as shown above, the magnitude values are multiplied by 2.

3
Figure 1: Magnitude Spectrum of $x(n)$

Figure 2: Phase Spectrum of $x(n)$
• If \( x(n) \) is periodic, the frequency resolution can only be improved by taking the FFT of several periods, and dividing by the number of periods. If \( x(n) \) is aperiodic, the frequency resolution of its FFT can be improved by adding zeros to the original finite length signal (i.e., padding with zeros). Resolution of periodic signals cannot be improved by adding zeros.

![Figure 3: Periodic signal \( x(n) \)](image)

1.2. Problems with Phase Calculations

The DFT values are generally complex,

\[
X(k) = |X(k)|e^{j\text{arg}[X(k)]} = X_R(k) + jX_I(k)
\]

where the phase is calculated according to

\[
\text{arg}[X(k)] = \arctan \left( \frac{X_I(k)}{X_R(k)} \right)
\]

except when \( X(k) \) is real. In that case, if \( X(k) > 0 \) then \( \text{arg}[X(k)] = 0 \) and whenever \( X(k) < 0 \) then \( \text{arg}[X(k)] = \pm \pi \). If \( X(k) = 0 \) then the phase is undefined (or of no importance). In plotting the magnitude and the phase spectra it should be remembered that the magnitude spectrum for real-valued signals \( x(n) \) is an even function of \( w \), while the phase spectrum is an odd function of \( w \).

If the corresponding z-transform \( X(z) \) of a period of \( x(n) \) has no singularities (poles or zeros) on the unit circle, then the phase \( \text{arg}[X(e^{jw})] \) is a continuous function of frequency. Even then, however, there are practical problems with the calculation of the phase. The following are some of them.
Figure 4: Magnitude spectrum of periodic signal

Figure 5: Magnitude spectrum of periodic signal (M=2)
Figure 6: Phase spectrum of periodic signal

Figure 7: Phase spectrum of periodic signal (M=2)
1. **Principal values** The function arctan(θ) is a periodic function of θ of period π and with discontinuities at multiples of ±π/2. Thus the principal values of the function are between (−π/2, π/2).

However, if one considers the signs of the real and the imaginary parts the principal values can be extended to (−π, π). For instance, the phase of 1 + j1 is π/4 and the phase of −1 − j1 is −3π/4 although the arctangent is not capable of differentiating the two values (it can only give the π/4 value).

**Problem #1** The phase values can only be calculated between −π and π. This problem is called the wrapping of the phase, i.e., whenever there are no singularities on the unit circle, the phase is continuous but when the phase value is outside [−π, π) it is reduced to an equivalent value between these bounds. This creates a fictitious discontinuity which is called the phase wrapping. It is possible given the wrapped phase to obtain the continuous or unwrapped phase in simply cases such as illustrated in the following example. In general, especially when the signal is contaminated with noise, phase unwrapping is a difficult task.

**Example** Consider the function with z-transform \( X(z) = z^{-10} \) or \( X(e^{jω}) = e^{-j10ω} \) and arg\( X(k) = -10ω \). The wrapped phase (Fig. 8) and the unwrapped phases (Fig. 9) are shown below. Notice that in this simple case, the unwrapped phase can be obtained from the wrapped phase by looking for jumps of 2π. In this case the magnitude is unity.

2. **Problem #2** The phase is only significant when the magnitude is significant, e.g., consider

\[
1 \times 10^{-6} + j1 \times 10^{-6} = 1.4142 \times 10^{-6} e^{jπ/4}
\]

although the phase is π/4 the magnitude of the above complex number is not significant and so the phase is not significant either.

2. **Filtering using the FFT**

Suppose \( x(n) \) is a signal of length \( N \) and is the input of a LTI filter with an impulse response \( h(n) \) of length \( M \). The output \( y(n) \) of the filter is the linear convolution (we will see later the reason for the name linear convolution, instead of simply convolution) and, as mentioned before, \( y(n) = [x * h](n) \) has a length \( M + N - 1 \). The DFT of \( y(n) \) is given by

\[
Y(k) = X(k)H(k)
\]  

(4)

Now, since \( y(n) \) has length \( M + N - 1 \) then \( Y(k) \) should be calculated for \( L \geq M + N - 1 \), and in order for the multiplication in (4) to make sense we need to have \( L \) values of \( X(k) \) and \( H(k) \). This implies that the FFT of \( x(n) \) and \( h(n) \) must be of length \( L \). If the constraints on the length of the FFTs are not followed the results are unpredictable, since it causes aliasing in the time domain. Once the \( Y(k) \) is obtained, by calculating the inverse FFT we’ll get \( y(n) \).

**Example**
Figure 8: Wrapped Phase

Figure 9: Unwrapped Phase
Fast convolution.
Suppose that the input \( x(n) \) and the impulse response \( h(n) \) of a LTI filter are

\[
x(n) = h(n) = \begin{bmatrix} 1 & 0 \leq n \leq 1 \\ 0 & \text{otherwise} \end{bmatrix}
\]

The output \( y(n) \) has a length 3, thus to calculate it using FFTs we need to calculate FFTs of length \( L \geq 3 \).

1. \( L = 4 \)

\[
X(k) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi}{4}nk} = 1 + e^{-j\pi k/2} = H(k)
\]

\[
Y(k) = X(k)H(k) = [1 + (-1)^k] + 2e^{-j\pi k/2} = \sum_{n=0}^{3} y(n)e^{-j\frac{2\pi}{4}nk}
\]

So that comparing term by term we get \( y(0) = 1, y(1) = 2, y(2) = 1, y(3) = 0 \) (this would be the result obtained using the inverse DFT). In this simple example we could easily show that the convolution (or linear convolution) gives the same results.

2. \( L = 3 \)

\[
X(k) = \sum_{n=0}^{2} x(n)e^{-j\frac{2\pi}{3}nk} = 1 + e^{-j2\pi k/3} = H(k)
\]

\[
Y(k) = X(k)H(k) = 1 + 2e^{-j2\pi k/3} + e^{-j2\pi 2k/3} = \sum_{n=0}^{3} y(n)e^{-j\frac{2\pi}{3}nk}
\]

So that \( y(0) = 1, y(1) = 2, y(2) = 1 \). Notice that in this case where \( L = 3 \) the results are the same as before, and coincide with the results of the linear convolution.

3. \( L = 2 \)

\[
X(k) = \sum_{n=0}^{1} x(n)e^{-j\frac{2\pi}{2}nk} = 1 + e^{-j\pi k} = H(k)
\]

\[
Y(k) = X(k)H(k) = 2 + 2e^{-j\pi k} = \sum_{n=0}^{1} y(n)e^{-j\frac{2\pi}{2}nk}
\]

So that \( y(0) = 2, y(1) = 2 \), which are not equal to the results obtained using the linear convolution.

2.1. Circular Convolution

When we implement the convolution \((x \ast h)(n)\) graphically, we need to reverse one of the signals and then shift linearly. The convolution calculation done with the FFT can be shown to be the result of doing a "circular" convolution, forced by the periodicity involved.
Rather than to develop the theory of the circular convolution let us illustrate the calculation using the signal $x(n)$ and the impulse response $h(n)$ used in the previous example. If we decide to take an FFT of length $L = 4$ of the signal $x(n)$, then theoretically that would be equivalent to creating a periodic signal $\tilde{x}(n)$ by repeating $x(n)$ periodically with period $L = 4$, and then calculating it Fourier series. Because of the periodic repetition, a periodic signal can be represented as a sequence of values of a period of $\tilde{x}(n)$ in a circle which repeat as we move around the circle (the starting point is not important, so it is taken as the the 0-degree point in the circle). Similarly we obtain a circular representation of $\tilde{h}(n)$, which is used to calculate the FFT of $h(n)$. Analogous to the linear convolution, we then need to fix one of the circular representations and to shift circularly the other. At each turn we calculate the product of opposite values on the circle and add to get the final value of the circular convolution at that particular value.

Doing this process we obtain $y(0) = 1, y(1) = 2, y(2) = 1, y(3) = 0$, which coincide with the results obtained before. Theoretically, the circular convolution $\oplus_L$ (notice that it depends on the length $L$, so that different results are obtained for different values of $L$) is defined as

$$\tilde{y}(n) = \sum_{k=0}^{L-1} \tilde{x}(k) \tilde{h}(n - k)$$

and then $y(n)$ is equal to a period of $\tilde{y}(n)$.

**Bottom Line:** If we wish to make the results of the linear convolution to coincide with that of the circular convolution (or equivalent the result obtained using the FFT and IFFT), i.e.,

$$y(n) = [x * h](n) = [\tilde{x} \oplus_L \tilde{h}](n) = \tilde{y}(n) \quad 0 \leq n \leq L - 1$$

(5)

then we must impose the condition that

$$L \geq \text{length of } x(n) + \text{length of } h(n) - 1$$

as the length of the circular convolution (or the FFT).

**Example**

Let's calculate the linear convolution of

$$x(n) = \begin{cases} 
1 & 0 \leq n \leq 7 \\
0 & \text{otherwise}
\end{cases}$$

with itself using circular convolution. We let $L$ (the length used for the FFT of $X(k)$) be first $12 < 8 + 8 - 1 = 15$ and as expected gives the wrong answer (Fig. 10). Using $L = 32$ we get the correct values for the linear convolution (Fig. 11). $L = 15$ is the minimum allowed length in order to get that the linear and the circular convolutions coincide.
Figure 10: Circular convolution (L=12)

Figure 11: Circular convolution (L=32)